On the pragmatics of modified numerals and modals

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1 Introduction

It has been known at least since Geurts and Nouwen (2007) that certain types of modified numerals give rise to so-called *ignorance effects*. As can be observed in (1), the numeral modifiers *at least* and *at most* are incompatible with exact knowledge, unlike *more than* and *fewer than*.

(1) a. I know exactly how many books there are in this bookshop, #and it's { at least / at most } 10,000.
b. I know exactly how many books there are in this bookshop, and it's { more than / fewer than } 10,000.

Following Nouwen (2010), I will refer to modified numerals that yield ignorance effects as *class B* modified numerals and to those that do not as *class A* modified numerals.

This paper is about what happens when class B modifiers occur with modals, as in (2)-(5).

(2) Jane is allowed to invite at most two friends.

(3) Jane is required to invite at least two friends.

(4) (?)Jane is required to invite at most two friends.

(5) ?Jane is allowed to invite at least two friends.

There are two things to note here. First, the combinations of *at most* with an existential modal and *at least* with a universal modal are felicitous, whereas the other two combinations are less so, with (5) being quite marked and (4) being slightly more natural, though still far less natural than (2), which appears to express the same meaning as (4). It should be noted that these judgments and other judgments throughout this paper are based on a context where the question under discussion is (6) rather than, for instance, (7).

(6) How many friends is Jane { allowed / required } to invite?

(7) Is Jane { allowed / required } to invite { at least / at most } two friends?

As has been found experimentally by Westera and Brasoveanu (2014), QUD matters greatly for the interpretation of modified numerals. Most of the theoretical literature on modified numerals, however, assumes a *how many* QUD, and I will stick with those types of cases here. Thus, the claim is that (5) is semantically deviant as an answer to the question in (6), (4) is dispreferred to (2) as an answer to (6), and both (2) and (3) are entirely felicitous.

The second observation is that the most prevalent reading of the natural combinations in (2) and (3) is one without ignorance: the so-called *authoritative* reading. (2) and (3) can be used to convey the number of friends Jane can or must invite without any ignorance on the part of the speaker. On this reading, (2) means that Jane obeys the rules if and only if she invites zero, one, or two friends. The relevant reading of (3) is that Jane obeys the rules if and only if she invites two or more friends.

Although less obvious, epistemic readings are also available for these sentences. (8) is designed to bring out this reading for (3).

(8) Jane's parents really want her to make more friends at her new school, so they told her she is required to invite some number of friends to her birthday party. I'm not sure how many friends Jane has to invite, but I know *she's required to invite at least two.*

In this example, the ignorance about what the minimum number of friends Jane is required to bring is. This minimum comes from the modal in the QUD, *How many friends is Jane required to invite?*, which presupposes that there is a lower bound to the number of friends Jane invites in all possible worlds. (8) says that this lower bound is at or above two.

A parallel context for (2) is given in (9). where the ignorance is about the maximum number of friends Jane is allowed to invite.

(9) Jane's parents want her to celebrate her birthday with only a tiny group of friends. I'm not sure exactly how many friends she can invite, but I know *she's allowed to invite at most two.*

Again, this maximum comes from the modal in the QUD. The question *How many friends is Jane allowed to invite?* indicates that there is an upper bound to the number of friends Jane can invite. This upper bound is then said to be at or below two.

Thus, (2) and (3) have ignorance readings, but the authoritative readings are far more salient. This contrasts with (4) and (5), where the ignorance reading is more prevalent. The ignorance reading of (4) is brought out in the context in (10).

(10) Jane's parents really want her to make more friends at her new school, so they told her she is required to invite some number of friends to her birthday party. I'm not sure how many friends Jane has to invite, but it's not that many: I know *she's required to invite at most two friends.*

The universal modal in the QUD *How many friends is Jane required to invite?* leads us to expect a lower bound. This lower bound, (10) says, is at or below the number two. Thus, despite the use of *at most two*,(10) is compatible with Jane being allowed to invite more than two people. (4) can also be used authoritatively to set an upper bound to the number of friends Jane is allowed to invite, but this is slightly odd as (2) is the preferred way to express this meaning.

Finally, the ignorance reading of (5) is illustrated in (11).

(11) Jane's parents want her to celebrate her birthday but they told her there's only allowed to invite a certain number of friends. I'm not sure exactly how many friends she can invite, but I know *she's allowed to invite at least two.*

The sentence containing the modified numeral can be thought of as answering the question *How many friends is Jane allowed to invite?*, which presupposes an upper bound. The speaker is not sure what this upper bound is, but she knows it is two or higher.

When we compare (2) to (4), the ignorance reading is more salient in (4) than in (2), whilst the authoritative reading is more prevalent in (2) than in (4). (5) is again the most extreme example: it lacks an authoritative reading and only gives rise to an ignorance reading given the *how many* QUD. Thus, the naturalness of the combination of modified numeral and modal correlates with its ability to obviate the ignorance inference and give rise to an authoritative reading instead. (2) and (3) are excellent ignorance obviators while (4) and (5) are less so, with (4) being better than (5).

As far as I am aware, no account of modified numerals has taken these two observations into account. Instead, all accounts on the market derive both ignorance and authoritative readings for all four combinations in (2)-(5) and predict that these examples are all equally felicitous (Geurts & Nouwen, 2007; Nouwen, 2010; Coppock & Brochhagen, 2013; Schwarz, 2013; Kennedy, 2015). In addition, existing accounts of these data derive readings for (2) that are too weak, as I will show in the next section.

In this paper I present an account that generates all and only the correct readings for sentences where modals interact with modified numerals. The account rests on a careful consideration of the scope facts and a mechanism of optional flattening in the framework of inquisitive semantics (e.g. Ciardelli, Groenendijk, & Roelofsen, 2018).

The paper is structured as follows. In the next section I will discuss previous accounts of modified numerals and modals and show that they do not capture the distinctions discussed above. Section 3 contains my account. In section 4 I discuss some further benefits of the account that go beyond the data in (2)-(5). Section 5 concludes.

2 Existing accounts of modified numerals and modals

Here I will show how existing accounts of class B modified numerals fail to capture the observations I made in the previous section as well as the strong reading that arises when *at most* is combined with an existential modal. The account I will discuss to make this point is Kennedy (2015), but other prominent accounts such as Coppock and Brochhagen (2013) and Schwarz (2013) face the same issues.

I will start by considering Kennedy's account of modified numerals combined with universal modals. For Kennedy, numeral modifiers are degree quantifiers that can undergo QR to take sentence scope. As such, they can take scope either above or below modals. It is this scope ambiguity that causes the pragmatic ambiguity for Kennedy: wide scope for the modified numeral yields ignorance readings, whereas narrow scope generates authoritative readings (a characteristic his account shares with e.g. Büring, 2008; Schwarz, 2013; Coppock & Brochhagen, 2013). These readings are derived through a scalar implicature mechanism. Kennedy's two denotations for (3) are given in (12), where *max* is an operator that picks out the member with the highest cardinality out of a set of degrees.

(3) Jane is required to invite at least two friends.

(12) a. $max\ \{n \mid \Box[\exists x[\text{invite}(\text{j},x) \wedge \text{friends}(x) \wedge \#x = n]]\} \geq 2$
b. $\Box\ [max\{n \mid \exists x[\text{invite}(\text{j},x) \wedge \text{friends}(x) \wedge \#x = n]\} \geq 2]$

(12-a) says that there is a set of numbers of friends Jane is required to invite, and the maximal member of this set is three or higher. (12-a) says that it is required that the maximum number of friends Jane invites is three or higher. These denotations are truth-conditionally equivalent and their truth conditions indeed correspond to the intuitive meaning of (3). However, they give rise to distinct scalar implicatures. For Kennedy, the alternatives to a sentence with a modified numeral are those where the numeral modifier is replaced by a different numeral modifier (*at least*, *at most*, *more than*, or *fewer than*) and those where the modified numeral as a whole is replaced by a bare numeral. Note that Kennedy's choice of alternatives indicates that he assumes a *how many* QUD. If he had taken the QUD to be (7), the only alternative would have been the negation of the relevant sentence. Given these alternatives, the relevant implicature of (12-a) are those schematised in (13), where K stands for *know*.

(13) $\{\neg K(max(\Box) = 2), \neg K(max(\Box) > 2)\}$.

Following the *standard recipe* for implicature calculation (Geurts, 2010; Sauerland, 2004), the implicatures in (13) are referred to as the *primary* implicatures of (3): the speaker does not know that the maximum required number is two and she also does not know that it is higher than two. These implicatures cannot be strengthened to the *secondary* implicatures given in (14): the speaker knows that the maximum is not two and she knows that it is not higher than two.

(14) $\{K\neg(max(\Box) = 2), K\neg(max(\Box) > 2)\}$.

The reason is that together, these secondary implicatures contradict the assertion in (12-a). Therefore, only the primary implicatures in (13) can be calculated, and this gives us the ignorance reading: the speaker knows that Jane is required to invite two or more friends, but she does not know whether the number of friends Jane must invite is two or higher.

For (12-b), the relevant primary implicatures are given in (15).

(15) $\{\neg K(\Box(max = 2)), \neg K(\Box(max > 2))\}$

Unlike before, the primary implicatures can now be strengthened to the secondary implicatures in (16) without contradicting the assertion.

(16) $\{K\neg(\Box(max = 2)), K\neg(\Box(max > 2))\}$

This is how we derive the authoritative reading: Jane is required to invite two or more friends, and, given that neither is necessary, both options of inviting two friends and inviting more than two friends are allowed. Thus, for (3), Kennedy's account correctly predicts the two readings we observe.

For (4), Kennedy derives the two meanings in (17).

(4) (?)Jane is required to invite at most two friends.

(17) a. $max\ \{n \mid \Box[\exists x[\text{invite(j,}x) \wedge \text{friends}(x) \wedge \#x = n]]\} \leq 2$
b. $\Box\ [max\{n \mid \exists x[\text{invite(j,}x) \wedge \text{friends}(x) \wedge \#x = n]\} \leq 2]$

The wide scope reading is that Jane is required to invite some number of friends, and this number is two or lower. Thus: she invites some minimum number of friends in every accessible world, and this number is two or lower. In the same way as before, we derive the primary ignorance implicatures in (18), which cannot be strengthened to secondary implicatures. Thus: the speaker does not know whether the number of friends Jane has to invite is two or less than two.

(18) $\{K\neg(\Box(max = 2)), K\neg(\Box(max < 2))\}$

Again as before, secondary implicatures can be derived for (17-b). The denotation in conjunction with the implicatures in (19) convey that in all accessible worlds, Jane invites a maximum of two friends. She cannot invite more than two, but she is free to choose between two and some number below two.

(19) $\{K\neg(\Box(max = 2)), K\neg(\Box(max < 2))\}$

In summary, the ignorance readings of both (3) and (4) can be said to partly answer the question *How many friends is Jane required to invite?*, which makes reference to some minimum number of friends Jane must invite. (3) then says that this minimum is at or above two, whilst (4) says that it is at or below two. The authoritative readings are that she is not allowed to invite less than two friends and that she is not allowed to invite more than two friends respectively. While these readings all seem to be attested, the fact that (4) is less natural-sounding than (3) is not taken into consideration, and neither is the fact that the ignorance reading of (4) is more prevalent than that of (3). These issues are relatively minor, but bigger problems occur when we consider cases with existential modals.

Let us begin by considering (5) and its denotations in (20).

(5) ?Jane is allowed to invite at least two friends.

(20) a. $max\ \{n \mid \Diamond[\exists x[\text{invite(j},x) \wedge \text{friends}(x) \wedge \#x = n]]\} \geq 2$
b. $\Diamond\ [max\{n \mid \exists x[\text{invite(j},x) \wedge \text{friends}(x) \wedge \#x = n]\} \geq 2]$

Exactly as above, we correctly derive an ignorance reading for (20-a). In conjunction with the primary implicatures in (21), the reading is that there is some maximum number of friends Jane can invite, and this number is two or higher; the speaker does not know which.

(21) $\{\neg K(max(\Diamond) = 2), \neg K(max(\Diamond) > 2)\}$

Unlike above, we derive a second set of ignorance implicatures for (20-b). The secondary implicatures in (23) contradict the assertion, so we are left with the primary implicatures in (22).

(22) $\{K\neg(\Diamond(max = 2)), K\neg(\Diamond(max > 2))\}$

(23) $\{K\neg(\Diamond(max = 2)), K\neg(\Diamond(max > 2))\}$

The result is the weak meaning that it is allowed for Jane to invite two or more friends, and the speaker is unsure whether two is allowed and whether more than two is allowed. It is unclear whether this weak ignorance reading is attested. Kennedy uses a mechanism with 'exhaustified alternatives' (Kratzer & Shimoyama, 2002) for these types of examples, which yields the authoritative reading given below.

(24) $\{K\neg(\Diamond(max = 2)) \wedge \neg(\Diamond(max > 2)),$
$K\neg(\Diamond(max > 2)) \wedge \neg(\Diamond(max = 2))\}$

Combined with the assertion, the reading is that it is allowed for Jane to invite two or more friends, and she can choose between inviting two friends and inviting more than two friends. Kennedy claims that this authoritative reading is attested in the text in (25) (p.35).

(25) Previously in Germany, students were allowed to take at least five years to complete the Magister's diploma, the basic university degree. But now, Germany has adopted the Anglo-Saxon style of bachelor's and master's degrees. The bachelor's degree is designed to take three years to complete; the master's, a further two years.

While (25) indeed has a reading without an ignorance inference, this is due to the plural subject *students*. It is known that plurals and universally quantified expressions, like modals, can obviate ignorance inferences. This has been observed by Nouwen (2010) and can be seen in his example in (26-a) (p.4) and its equivalent with a universal quantifier in (26-b).

(26) a. Computers of this kind have at least 512MB of memory.
b. All computers of this kind have at least 512MB of memory.

The most salient reading of the examples in (26) is one without ignorance. Instead, the sentences convey that computers vary in their memory capacity, but no computer has a memory capacity of less than 512MB. The same is true for (25), where students vary in the number of years they take to complete the diploma. Kennedy mentions in a footnote (fn. 7) that Doris Penka has pointed out this issue to him, but he does not seek to resolve it. When we consider a case without a plural and the *How many* QUD, only an ignorance reading arises, as (27) demonstrates.

(27) Q: How much time is Cara allowed to take to complete her degree?
A: Cara is allowed to take at least five years.

Thus, Kennedy incorrectly predicts authoritative readings for cases like (5).

Finally, let us return to (2), for which Kennedy generates the two sets of truth conditions in (28).

(2) Jane is allowed to invite at most two friends.

(28) a. $max\ \{n \mid \Diamond[\exists x[\text{invite(j,}x\text{)} \wedge \text{friends(}x\text{)} \wedge \#x = n]]\} \leq 2$
b. $\Diamond\ [max\{n \mid \exists x[\text{invite(j,}x\text{)} \wedge \text{friends(}x\text{)} \wedge \#x = n]\} \leq 2]$

As with (5), the ignorance reading is correctly generated for (28-a). The primary implicatures in (29) cannot be strengthened to the secondary implicatures in (30) without this resulting in a contradiction to the assertion, and so we get the ignorance reading that there is some upper bound to the number of friends Jane can invite, and this upper bound is at or below two.

(29) $\{\neg K(max(\Diamond) = 2), \neg K(max(\Diamond) < 2)\}$.

(30) $\{K\neg(max(\Diamond) = 2), K\neg(max(\Diamond) < 2)\}$.

But (28-b) cannot give us the authoritative reading. First, as with (5), we cannot calculate secondary implicatures, given in (32), as this would again yield a contradiction. Only the primary implicatures in (31) can be generated.

(31) $\{K\neg(\Diamond(max = 2)), K\neg(\Diamond(max < 2))\}$

(32) $\{K\neg(\Diamond(max = 2)), K\neg(\Diamond(max < 2))\}$

Thus, only an ignorance reading is derived. Second, (28-b) only states that there is a permissible world where Jane invites zero, one, or two friends, which is entirely compatible with her inviting more than two friends in other worlds.

The authoritative reading we observe for (2) is far stronger than this: it does not allow Jane to invite more than two friends.

Kennedy remedies the first issue by using exhaustified alternatives, as he did for (5). These exhaustified alternatives are given below and they do indeed yield an authoritative reading.

(33) $\{K\neg(\Diamond(max = 2)) \wedge \neg(\Diamond(max < 2)),$
$K\neg(\Diamond(max < 2)) \wedge \neg(\Diamond(max = 2))\}$

The second issue, however, is difficult to resolve. Kennedy claims that an implicature mechanism is responsible for the upper bound of (2), but the central characteristic of an implicature is cancellability. Therefore, (34-a). with the modifier *fewer than*, can be said to have an upper bound implicature, but (34-b), with *at most*, cannot.[1]

(34) a. Jane is allowed to invite fewer than two friends, but more is fine too.
b. Jane is allowed to invite at most two friends, #but more is fine too.

Thus, there are two main issues with Kennedy's analysis. The first is that he

[1] Kennedy gives two arguments in favour of his implicature story. First, according to Kennedy, (i), like (34-b), sounds bad, and this is clearly a case of a scalar implicature creating an upper bound. I am not convinced that this judgment is right, and even Kennedy himself only claims that it's 'more like' (34-b) than like (34-a) rather than exactly like (34-b).

(i) Jane is allowed to invite zero to three friends, but more is fine too.

Additionally, if Kennedy is correct we would expect (ii) to sound as bad as (34-b), but it is actually quite felicitous.

(ii) Jane is allowed to invite between zero and three friends, but more is fine too.

Second, Kennedy gives the example from a job advert in (iii) (p.34), which he indicates has the scope configuration $\Box > \Diamond > max$. This, he argues, is a case where *at most* does not set an upper bound, because employees who are able to lift more than 70lbs will obviously not be turned down for the job.

(iii) [...] Must be able to lift at most 70lbs.

Aside from the fact that it is unknown what exactly the effect of stacked modals on modified numerals is, I would argue that (iii) may actually have the scope configuration $max > \Box\Diamond$. Kennedy rules out this scope configurations because in his theory, giving *at most* wide scope yields ignorance inferences. However, this is merely an assumption, and as will become clear in the next section, my theory does not have this characteristic. Furthermore, this case is not captured by Kennedy's theory as the QUD of (iii) is not a *how many* question. The interviewer's answer in (iv) is rather strange because it has an ignorance inference. The yes/no QUD in (v) appears to be more natural. This case is thus separate from the other ones we have discussed and cannot be directly compared to them.

(iv) Interviewee: How much do I have to be able to lift to get this job?
Interviewer: ??You must be able to lift at most 70lbs.

(v) Interviewee: Should I be able to lift up to 70lbs to get this job?
Interviewer: (?)Yes, you must be able to lift at most 70lbs.

derives both ignorance and authoritative readings for all four possible combinations of modified numerals and modals, not taking into account the distinction between natural and unnatural combinations and the fact that authoritative readings are far more salient for the natural combinations than for the unnatural ones, and even absent for the case where *at least* occurs with an existential modal. The second is that he is unable to derive the strong upper-bounded reading for the combination of *at most* with an existential modal, even though this is the most prevalent reading of such sentences.

I have chosen to discuss Kennedy's account here but these issues also hold for other accounts that use scalar implicatures to generate ignorance inferences (such as Büring, 2008 and Schwarz, 2013) as well as Coppock and Brochhagen (2013), which uses a quality implicature system to obtain ignorance readings. While there are a few authors who do derive the correct readings for cases with *at most* and an existential modal (Nouwen, 2010; Penka, 2014), no-one has observed the dichotomy between the *at least* - $\Box$ and *at most* - $\Diamond$ pairs on the one hand and the *at least* - $\Diamond$ and *at most* - $\Box$ pairs on the other. In the next section I will present an analysis that resolves these two issues.

3 Analysis

Below I will outline an analysis that generates the readings we observe for all four combinations in (2)-(5). Because I have borrowed the implicature generation mechanism I use from Ciardelli et al. (2018), I first briefly go over this method in the next section. In section 3.2 I discuss how the lexical entries I propose can be used to calculate ignorance implicatures in cases where modified numerals occur without modals. Section 3.3 contains the mechanism I use to derive two readings for those cases where modified numerals occur with modals. Section 3.4 contains an account of the two natural combinations of modified numerals and modals, and section 3.5 is about the less natural ones.

3.1 Inquisitive semantics and epistemic inferences

We have seen that Kennedy (2015), among others, uses the standard recipe to obtain ignorance implicatures. I will instead use a mechanism in the inquisitive semantics framework to calculate these inferences. This mechanism generates ignorance implicatures as quality implicatures rather than quantity implicatures. In this section I will briefly outline the basic notions of inquisitive semantics and the implicature calculation mechanism that we will need.

In inquisitive semantics, propositions are sets of possibilities. A possibility is what is usually called a proposition: an expression of type p $(=\langle s,t\rangle)$. When a proposition contains more than one possibility, it is called an *inquisitive* proposition. By using an inquisitive proposition, a speaker is said to raise an issue. An issue is a request for the hearer to resolve the issue: to pick out the possibility in the proposition that is true. For example, the disjunction in (35) raises two possibilities: the possibility that Anne speaks French and the possibility that she speaks German.

(35) Anne speaks French or German.

This is represented in (36), where p_f stands for the set of worlds where she speaks French and p_g stands for the set of worlds where she speaks German.

(36) $\{p_f, p_g\}$

Inquisitive semantics has been used in the literature on modified numerals to calculate epistemic inferences (Coppock & Brochhagen, 2013; Blok, 2015, 2016; Ciardelli, Coppock, & Roelofsen, 2016; Cremers, Coppock, Dotlacil, & Roelofsen, 2017; Blok, 2019). In this literature, the epistemic inferences class B numeral modifiers give rise to are said to be quality implicatures. I will follow Ciardelli et al. (2016)'s way of calculating these implicatures, which is slightly different from the way it is done in the other literature I have cited.[2] The idea is that the Gricean Maxim of Quality (Grice, 1975) consists of two parts, given below.

(37) The Maxim of Quantity in Inquisitive Semantics (Ciardelli et al., 2016)
  a. $s \subseteq \text{info}(\phi)$
  b. if ϕ is inquisitive, then $s \notin [\![\phi]\!]$

The Maxim of Quality is only satisfied if both conditions are met. info(ϕ) is the information contained in a proposition ϕ: the union of all its possibilities. s is the speaker's information state: a set of worlds. (37-a) says that the speaker's information state must be a subset of the informative content of the proposition she utters. This is the original Quality Maxim: say only what you believe to be true. The second part only comes into play when an inquisitive proposition is used. In this case, the speaker's information state cannot be an element of the proposition she utters. That is, none of the possibilities in the proposition can be the speaker's information state. For instance, say that the speaker knows that Anne speaks French: the possibility p_f is her information state. Then if she utters (35), she violates the Maxim of Quality, because she raises a proposition that contains a possibility that is her information state. In less technical terms, she suggests multiple options, inviting the hearer to select the true option, even though she herself knows which option is true.[3]

When class B modified numerals are used without a modal, they give rise to epistemic inferences. To derive these inferences, Ciardelli et al. (2016) propose

[2]Specifically, Ciardelli et al. (2016)'s theory is formulated in the framework Inq_B, which is the downward closed version of inquisitive semantics: whenever a possibility occurs in a proposition, all of its subsets are also in the proposition. The other papers cited here are formulated in $\text{Inq}_\cup$, which is not downward closed and therefore allows nested possibilities: one possibility can be contained in another possibility in the same proposition in a non-trivial way.

[3]Of course this is a simplified model: any speaker's information state will contain more information than just the information that Anne speaks French.

that class B numeral modifiers such at *at least* and *at most* give rise to inquisitive propositions containing two possibilities. For instance, (38) contains the two possibilities illustrated in (39): the possibility that Anne speaks exactly two languages, represented by p_2, and the possibility that she speaks three or more languages, represented by $p_{[3-\infty)}$.

(38) Anne speaks at least two languages.

(39) $\{p_2, p_{[3-\infty)}\}$

Assuming that the speaker is being cooperative and following the Quality Maxim, we can conclude from the fact that she used an inquisitive proposition that she does not know which of the possibilities in the proposition are true. Thus, she does not know if Anne speaks at least two languages and she does not know if Anne speaks three or more languages. This is how the epistemic implicature comes about. I will use this method in my account of the class B numeral modifiers *at least* and *at most*.

3.2 The basics

I assume the syntactic structure in (40) for sentences like (2)-(5).

(40) [$\Box$/$\Diamond$ [Jane invites [{ at least / at most } [[two *many*] friends]]]]

Here *many* is Hackl's (2000) counting quantifier, defined below, which turns *two friends* into a regular quantifier.

(41) $[\![many]\!] = \lambda d_d \lambda P_{\langle e,t\rangle} \lambda Q_{\langle e,t\rangle}.\exists x[\#x = d \wedge P(x) \wedge Q(x)]$

I take *at least* and *at most* to be focus-sensitive expressions that can associate not only with numerals but with with any element in their c-command domain. This behaviour is illustrated in (42), where *at least* associates with *three biscuits* rather than *three*.

(42) Hakim ate at least [three BISCUITS]$_F$, and he probably had other things too.

The structure in (40) is not the standard structure assumed for sentences with modified numerals. Since Hackl (2000), it has become commonplace to assume the structure of the DP given in (43) (Hackl, 2000; Nouwen, 2010; Schwarz, 2013; Coppock & Brochhagen, 2013; Kennedy, 2015).

(43) [[at least [2 *many*]] friends]

In (43), however, *at least* does not c-command *friends* and cannot associate with it, excluding cases like (42). I believe that the DP structure in (40), also assumed by Krifka (1999) and Geurts and Nouwen (2007).

Furthermore, assume *at least* and *at most* to be of type $\langle p, p\rangle$, where p is short for the inquisitive propositional type $\langle\langle s,t\rangle, t\rangle$. As such, we can assume that they can occur at a position above or below the modal at LF, as shown in (44) (this will be refined below).

(44) [{ at least / at most } [$\Box/\Diamond$ [{ at least / at most } [Jane invites [[two *many*] friends]]]]]

Recall that the QUD of an expression containing a modified numeral is essential for the calculation of its implicatures. The link between focus marking and the QUD is made by the Focus Principle, given in (45) (Beaver & Clark, 2008). What this principle comes down to for our current purposes is that the QUD must be a subset of the Rooth-Hamblin alternatives of an expression (Hamblin, 1971; Rooth, 1985, 1992).

(45) **Focus Principle**

a. Some part of a declarative utterance should give an answer to the CQ

b. If Q is a set of Rooth-Hamblin alternatives, and A is a natural language expression, then A gives an answer to Q if the focus value of A is a subset of Q

Furthermore, I assume that there exists an ordered version of the QUD, S. This ordering can be an entailment ordering or a pragmatic ranking. For instance, for (46) the QUD in (47) can be ranked pragmatically as in (48).

(46) Laura is an [associate professor]$_F$.

(47) QUD = { Laura is an assistant professor, Laura is an associate professor, Laura is a full professor }

(48) S = { ⟨Laura is a full professor, Laura is an associate professor⟩, ⟨Laura is an associate professor, Laura is an assistant professor⟩, ... }

I propose the lexical entries for *at least* and *at most* given below, with $\textsc{max}_{AL}$ as defined in (51).[4]

(49) $[\![\text{at least}]\!]^{S,AL} = \{\lambda p.\textsc{max}_{AL}\ p\ ,\ \lambda p. \cup \{\textsc{max}_{AL}\ p' \mid p' >_S p\}\}$

(50) $[\![\text{at most}]\!]^{S,AL} = \{\lambda p.\textsc{max}_{AL}\ p\ ,\ \lambda p. \cup \{\textsc{max}_{AL}\ p' \mid p' <_S p\}\}$

(51) $\textsc{max}_{AL} = \lambda p.\{w | w \in p \wedge \neg\exists p'[p' >_{AL} p \wedge w \in p']\}$

A quick look at these lexical entries reveals that they are disjunctive, taking the union of two possibilities. It is this disjunctive nature that will give us the epistemic readings that we need.

To understand these lexical entries, we must first understand what $\textsc{max}_{AL}$ does. $\textsc{max}_{AL}$ takes a possibility and returns a subset of this possibility. None of the worlds in this subset are in a possibility p' that is ranked higher than p. This ranking is dependent on an ordered set of alternatives AL. AL is usually equivalent to S but comes about in a different way. I will have more to say

[4] When not specified, all denotations represent the ordinary semantic value rather than the set of alternatives. I will only use the superscripts O and A when the distinction is relevant.

about this distinction in the next section. For now, we can simply assume $AL = S$.

At least takes a possibility or set of worlds as an argument. It returns a proposition containing two possibilities. The first is the possibility that results from applying $\textsc{max}_{AL}$ to the possibility it takes as an argument: $\textsc{max}_{AL}\ p$. The second is the union of the set of possibilities $\textsc{max}_{AL}\ p'$ for all possibilities p' that are ranked higher on S than p. *At most* also returns $\textsc{max}_{AL}\ p$ and it returns the set of possibilities $\textsc{max}_{AL}\ p'$ for all possibilities p' that are ranked lower on S than p.

Let us consider the example in (52) to make this more concrete. The prejacent of *at least* is the singleton proposition in (53).

(52) Abdullah ate at least two sandwiches.

(53) $[\![\text{Abdullah ate two sandwiches}]\!]^O = p_2 = \{\exists x[\#x = 2 \wedge \text{sandwiches}(x) \wedge \text{ate}(\text{Abdullah}, x)]\}$

If the relevant scale is provided by the numeral, then the set S is ordered as in (54), where p_n stands for $[\![\text{Abdullah ate } n \text{ sandwiches}]\!]^O$. The set AL used by $\textsc{max}_{AL}$ is equivalent to S. I assume a one-sided meaning of bare numerals. This means that p_n contains the world $w_{[n]}$ and also the worlds $w_{[n]+[1]}$, $w_{[n]+[2]}$, etc., where $[n]$ means 'exactly n'.

(54) $S = AL = p_0 < p_1 < p_2 < p_3 < p_4 \ldots$

At least takes this possibility as an argument through pointwise functional application (or Hamblin functional application, Hamblin, 1973) and returns (55).

(55) $\{p_2 \wedge \neg p_3, p_3\}$

(55) contains two possibilities. The first possibility, $p_2 \wedge \neg p_3$, is the result of applying $\textsc{max}_{AL}$ to the prejacent p_2. $\textsc{max}_{AL}$ takes the possibility p_2 and takes out all the worlds that are in some alternative ordered higher than p_2. Thus, it takes out all the worlds that are in p_3. Assuming a discrete scale, these are the worlds $w_{[2]}$, $w_{[3]}$, $w_{[4]} \ldots \infty$. Thus, it takes out all the worlds in p_3. It also takes out all the worlds that are in p_4 but not in p_2, but this is vacuous, as these worlds are also present in p_3. Thus, the result is $p_2 \wedge \neg p_3$.

The second possibility is the result of applying $\textsc{max}_{AL}$ to all possibilities p' in S that are ordered higher than p. This process is shown in (56).

(56) $\textsc{max}_{AL}\ p_3 = p_3 \wedge \neg p_4$,
$\textsc{max}_{AL}\ p_4 = p_4 \wedge \neg p_5$,
$\textsc{max}_{AL}\ p_5 = p_5 \wedge \neg p_6$,
etc.

As the denotation of *at least* shows, after the application of $\textsc{max}_{AL}$ we take the union of the set of all the possibilities in (56). This union is simply p_3. Therefore, p_3 is the second possibility in (55).

Thus, the sentence in (52) conveys two possibilities: the possibility that

Abdullah ate exactly two sandwiches and the possibility that he ate three or more sandwiches. The Inquisitive Maxim of Quality says that one should not utter an inquisitive proposition unless all the possibilities in the proposition are live possibilities in your information state s. Thus, neither the possibility $p_2 \wedge \neg p_3$ nor the possibility p_3 are in the information state of the speaker of (52). Therefore, the speaker is ignorant about which of the possibilities are true. This is how the epistemic inference is derived.

The derivation of a sentence with *at most*, like (57), is very similar. The prejacent of *at most*, like the prejacent of the *at least* sentence, is p_2, spelled out in (53).

(57) Abdullah ate at most two sandwiches.

Applying *at most* to the prejacent derives the proposition in (58).

(58) $\{p_2 \wedge \neg p_3, \{w_{[0]}, w_{[1]}\}\}$

First we get $p_2 \wedge \neg p_3$ from $\text{MAX}_{AL}\ p_2$, as above. Then we apply MAX_{AL} to all propositions that are ordered lower than p_2 on S, as in (59).

(59) $\text{MAX}_{AL}\ p_0 = p_0 \wedge \neg p_1,$
$\text{MAX}_{AL}\ p_1 = p_1 \wedge \neg p_2$

The union of these two possibilities is $\{w_{[0]}, w_{[1]}\}$.[5] (57) then conveys that either Abdullah ate exactly two sandwiches or he ate fewer than two. As the proposition is inquisitive, an epistemic implicature is derived through the Maxim of Quality.

I have assumed here that *at least* and *at most* take possibilities as arguments. In reality, I believe that their type is flexible. One way to implement this is to follow Coppock and Brochhagen (2013) and assume flexible lexical entries, as in (60)-(61), where α stands for any type ending in p and β is whatever type α takes as an argument.

(60) $[\![\text{at least}]\!]^{S,AL} = \{\lambda\alpha\lambda\beta.\text{MAX}_{AL}\ (\alpha(\beta))\ ,\ \lambda\alpha\lambda\beta. \cup \{\text{MAX}_{AL}\ p' \mid p' >_S \alpha(\beta)\}\}$

(61) $[\![\text{at most}]\!]^{S,AL} = \{\lambda\alpha\lambda\beta.\text{MAX}_{AL}\ (\alpha(\beta))\ ,\ \lambda\alpha\lambda\beta. \cup \{\text{MAX}_{AL}\ p' \mid p' <_S \alpha(\beta)\}\}$

This way, *at least* and *at most* can be interpreted *in situ*. For instance, in (62) *at least* takes *five beers* as an argument, which is turned into a regular quantifier over individuals through Hackl's (2000) *many* quantifier. The definition of *five* many *beers* is given in (63).

(62) Maxine drank at least five beers.

(63) $[\![\text{five } \textit{many} \text{ beers}]\!] = \{\ \lambda P_{\langle e,p\rangle}.\exists x_e[\#x = 5 \wedge \text{beers}(x) \wedge P(x)]\ \}$

The relevant denotation of *at least* is the one in (64). Applying *at least* to *five*

[5] This is equivalent to $\neg p_2$.

beers yields (65), which can then be combined with *drank* using QR or type shifting.

(64) $[\![\text{at least}]\!]^{S,AL} = \{\lambda\mathcal{P}_{\langle\langle e,p\rangle,p\rangle}\lambda Q_{\langle e,p\rangle}.\text{MAX}_{AL}\left(\mathcal{P}(Q)\right), \lambda\mathcal{P}_{\langle\langle e,p\rangle,p\rangle}\lambda Q_{\langle e,p\rangle}.\cup\{\text{MAX}_{AL}\ p' \mid p' >_S \mathcal{P}(Q)\}\}$

(65) $[\![\text{at least five } \textit{many} \text{ beers}]\!] = \{\lambda Q_{\langle e,p\rangle}.\text{MAX}_{AL}\left(\exists x_e[\#x = 5 \wedge \text{beers}(x) \wedge Q(x)]\right), \lambda Q_{\langle e,p\rangle}.\cup\{\text{MAX}_{AL}\ p' \mid p' >_S \exists x_e[\#x = 5 \wedge \text{beers}(x) \wedge Q(x)]\}\}$

Throughout the rest of this paper I will continue to use the denotations in (49)-(50) because this makes for less complex derivations. Provided that the order of interpretation stays the same when there are different operators in the sentence, it makes no difference for the end result.

So far the readings I have derived are the same as those in Ciardelli et al. (2016), although the compositional implementation of the idea is my own. In the next section I will show how variation and epistemic readings with class B modifiers and modals can be derived using the lexical entries I proposed above in combination with an additional mechanism of optional flattening that has not been used in the literature.

3.3 The mechanism

My account rests on two assumptions. The first is that class B numeral modifiers must take scope over existential modals. The second is that modals optionally flatten the alternatives in their scope. Here I will discuss each of these assumptions.

3.3.1 Class B modifiers and scope

There are three arguments why class B numeral modifiers must outscope existential modals. The first argument comes from the data presented in section 2: weak readings where existential modals take scope over *at most* simply do not exist. The fact that the upper bound of sentences like (2) cannot be cancelled signals that an implicature mechanism cannot be responsible for it. Therefore, the only way to get the strong reading we need is by letting *at most* take wide scope.

The second argument for the claim that class B numeral modifiers must take scope over existential modals is that, on closer inspection of the judgments, we can see that the surface scope reading is also not attested in the *at least* case. Consider (66) and the two theoretically possible readings in (67).

(66) Marin is allowed to read at least five books.

(67) a. $\Diamond\ [\ \textit{max}\ \{\ n \mid \text{Marin reads } n \text{ books}\ \} \geq 5\ ]$
b. $\textit{max}\ \{\ n \mid \Diamond\ [\ \text{Marin reads } n \text{ books}\ ]\ \} \geq 5$

Semantically, the difference in meaning between the surface scope reading in (67-a) and the inverse scope reading in (67-b) is that only the inverse scope reading carries the presupposition that there is an upper bound to what is allowed. Thus, (67-a) merely conveys that there is a permissible world where Marin reads five or more books. (67-b) expresses that there is a maximum number of books Marin is allowed to read, and that maximum is five or higher. This presupposition that there is a maximum allowed number follows from the semantics of *max*. *max* picks the highest number out of the set, but if there is no highest number, this is not possible. In this case, *max* cannot pick any number. We get $? \geq 5$, and there is no way for us to test whether the sentence is true. Therefore, the only way for the sentence to receive a truth value is if there is a maximum number of books Marin is allowed to read.

Let us consider our judgments on (66) and compare them to our judgments on (68). (68) contains the class A modifier *more than* and is therefore predicted to have a surface scope reading, unlike (66). This is because *more than* is not a class B numeral modifier, and the claim I make here is that only class B numeral modifiers must outscope existential modals.

(68) Marin is allowed to read more than five books.

Intuitively, (66) is felicitous only in a situation where there is indeed some maximal number of books Marin can read, though the speaker does not know what this upper bound is. (68), on the other hand, does not seem to carry such a requirement. This indicates that the inverse scope reading is the only possible reading for (66). Just like sentences with expressions like *at most* and an existential modal, expressions like *at least* must take scope over the modal. The surface scope reading is not attested.

To clarify the intuition, let us consider the following scenario. Say Marin is a school child in a school with the following rules. The children in Year 4 are allowed to read as many books as they like during the school year. The children in Year 5, on the other hand, are expected to focus more on subjects such as maths and geography, and they have an upper limit to the number of books they can read at school. The exact upper limit varies from child to child and depends on the child's reading level and the child's grades for other subjects. In addition, new research has just been published that indicates that children who read 20 books a year or more have better vocabulary than those who read fewer than twenty books a year. One day, Marin's dad and another parent, David are talking about this new research. David wonders about Marin's vocabulary and asks the question in (69).

(69) David: Is Marin in Year 4 or in Year 5?

Let us first say that Marin is in year 5. This means that there is a limit to the number of books she is allowed to read. Fortunately for her vocabulary, the limit in Marin's particular case is 25, so it is higher than twenty. In this case, Marin's father can felicitously utter either (70-a) or (70-b). (70-a) presupposes that there is an upper bound to the number of books Marin can read, and as this is indeed the case, the utterance is unproblematic.

(70) a. Marin is in Year 5. She is allowed to read at least twenty books.

b. Marin is in Year 5. She is allowed to read more than twenty books.

Now let us consider a scenario where Marin is in Year 4. This means that Marin can read as many books as she wishes. In this case, only (71-b) is a felicitous answer to David's question. In (71-b), Marin's father states that Marin is in Year 4, and as a result, she is allowed to read more than 20 books, which is good for her vocabulary. When we replace *more than* with *at least*, as in (71-a), the statement is no longer felicitous in the context. Given that Marin can read an unlimited number of books, it is very odd to state that she can read 'at least 20' books.

(71) a. Marin is in Year 4. #She is allowed to read at least twenty books.
b. Marin is in Year 4. She is allowed to read more than twenty books.

If (71-a) were ambiguous between a surface scope reading and an inverse scope reading, it would be felicitous in the given context. Marin's father could then simply have intended the surface scope variation of the sentence. The proposition corresponding to the surface scope configuration does not carry the presupposition that there is an upper limit to what is allowed. Therefore, it should be as good as (71-b) in this scenario. The fact that it is not felicitous shows that the surface scope reading does not exist. Therefore, the surface scope LF that would yield this reading must not exist, either. Though the truth-conditional difference is harder to detect in this case, it is nevertheless real, and we can conclude that *at least*, like *at most*, must outscope existential modals.[6]

[6]At this point the reader may wonder if the same point can be made for interactions between *at least* and universal modals such as (i). The two possible denotations of (i) are given in (ii), with (ii-a) being the surface scope reading and (ii-b) being the split reading.

(i) Marin has to read at least five books.

(ii) a. $\Box$ [$\textsc{max}$ $\{n \mid \exists x \wedge \#x = n \wedge \text{book}(x) \wedge \text{read(Marin}, x)\} \geq 5]$
b. $\textsc{max}$ $\{n \mid \Box[\exists x \wedge \#x = n \wedge \text{book}(x) \wedge \text{read(Marin}, x)]\} \geq 5$

These two denotations are truth-conditionally equivalent: both are true if and only if the lowest number of books Marin reads in all deontically accessible worlds is five. As in the case with existential modals, (ii-b) carries the presupposition that there is an upper bound. Here the presupposition is that there is an upper bound on what is required. Thus, there is a point where the number of books Marin has read is sufficient. That means that the only way the presupposition can fail is in a context in which it is known that the requirement has no limit. One scenario in which it is known that the requirement is infinite is in the myth of Sisyphus, who has been condemned to roll a rock up the hill only to watch it roll back down, repeating this action for eternity. To simplify the example, let us assume that the myth is slightly different: Sisyphus has an infinite number of rocks at his disposal, and he has to roll them all up the hill, only to watch them roll back down on the other side of the hill. Then (iii) should lead to a presupposition failure under the split reading. The presupposition is that there is an upper bound to what is required, but in this case, there is no such upper bound.

(iii) #Sisyphus must roll at least a million rocks up the hill.

This sentence does indeed sound quite bad in the given context, and it appears to be worse than (iv), where *more than* is used instead of *at least*.

(iv) ?Sisyphus must roll more than a million rocks up the hill.

The third and final argument I will present for the statement that class B numeral modifiers must take scope over existential modals is a syntactic argument. Let us turn to (72) and (73). In these sentences, *more than* and *at least* occur in a finite clause under an modal. Finite clauses are known to be islands for QR (e.g. Reinhart, 2006; Fox, 2000 and references cited therein; see also Büring, 2008 for an example with a finite clause island for degree QR). This means that in these examples, the relevant expressions are stuck below the modal. If class B numeral modifiers must outscope existential modals, this means that forcing them to be in the scope of existential modals is predicted to result in infelicity.

This prediction is borne out. As a baseline, consider (72) with a universal modal. Here, no problems arise: both the class A modifier *more than* and the class B modifiers *at least* can be used. When we consider similar cases with existential modals, however, we see a contrast. *More than* but not *at least* sits happily in the scope of the existential modal. Forcing *at least* under the modal leads to an infelicitous sentence. This indicates that *at least* cannot scope below an existential modal; it to scope above it. When this is not possible, the derivation crashes.[7]

(72) a. The government requires that organic chickens have more than 1000 cm^2 of space.
b. The government requires that organic chickens have at least 1000 cm^2 of space.

However, we know that *at least* gives rise to epistemic inferences, and this may interfere with our judgments. If the speaker knows that there is no upper bound to the number of rocks Sisyphus must roll up the hill, then she is not ignorant about the precise number under discussion. This may explain the contrast between (iii) and (iv). In addition, both sentences also suffer from a Quantity violation: why say that Sisyphus must roll a million rocks or more up the hill when you know that the amount is in fact infinite? Therefore, their oddness cannot be attributed to presupposition failure. We could change the examples to (v) to remedy this, but as *at least* is semantically vacuous in *at least an infinite number* and *more than an infinite number* is impossible, these sentences are also bad for independent reasons.

(v) a. #Sisyphus must roll at least an infinite number of rocks up the hill.
b. #Sisyphus must roll more than an infinite number of rocks up the hill.

In other words, it seems to me that it is impossible to test whether *at least* can outscope a universal modal in this way. However, as will become clear in this section, there is a syntactic test that indicates that *at least* can in fact take scope under universal modals.

[7] The sentence in (73-b) improves considerably in an echoic context, such as in the dialogue in (i).

(i) A: Are nurses allowed to work at least 40 hours a week?
B: Yes, new government regulations allow that nurses work { at least / minimally } 40 hours a week.

However, echoic contexts are not a good test for grammaticality or felicity in that they tend to allow things that are normally ungrammatical. For instance, the PPI *someone* is licensed under negation in the dialogue in (ii).

(ii) A: John saw someone.
B: No, he didn't see someone.

(73) a. New government regulations allow that nurses work more than 40 hours a week.

b. #New government regulations allow that nurses work at least 40 hours a week.

(74) and (75) make the same point for downward entailing modified numerals: the class A modifier *fewer/less than* does not mind taking scope under a universal modal or an existential modal. The class B modifier *at most*, on the other hand, only wants to take scope under a universal modal, and resists staying under an existential modal.

(74) a. The factory farm requires that the chickens have less than 1000 cm^2 of space.

b. The factory farm requires that the chickens have at most 1000 cm^2 of space.

(75) a. New government regulations allow that nurses work fewer than 40 hours a week.

b. #New government regulations allow that nurses work at most 40 hours a week.

I will take the structure where the numeral modifier takes wide scope to be the basic structure from which I derive most readings. In the case of existential modals, this structure coincides with the only possible scope configuration. The case with universal modals is slightly different. I will return to this point in section 4.1.

3.3.2 Optional flattening

The second assumption I make has to do with the role of the modal. Coppock and Brochhagen (2013), inspired by Kratzer and Shimoyama (2002), assume that a modal flattens the set of possibilities in its scope. That is, when the prejacent of a modal contains multiple possibilities, the modal returns the union of these possibilities. For them, this mechanism is linked to scope. They assume that all possible scope configurations between modified numerals and modals are possible, and the modal can flatten only when it takes wide scope.

To see how this mechanism works, consider a sentence where a universal modal occurs with *at least*. When *at least* takes wide scope, the denotation Coppock and Brochhagen derive is as in (76).[8]

(76) *at least 2* $> \Box \rightarrow \{\Box p_2, \Box p_3, \Box p_4, \ldots\}$

This denotation is inquisitive, and therefore an epistemic inference is derived: the speaker is unsure about whether two is required, whether three is required, etc.

[8] Coppock and Brochhagen's semantics of *at least* and *at most* is very different from mine. I use their analysis here purely to illustrate how flattening works.

When the modal takes wide scope, however, it flattens the possibilities created by *at least*, returning a single possibility, as in (77). This denotation is no longer inquisitive, and as a result, no epistemic inference is derived.

(77) $\Box >$ *at least 2* $\rightarrow \{\Box \cup \{p_2, p_3, p_4, \ldots\}\} = \{\Box p_2\}$

The way I use the flattening mechanism is different. As mentioned above, *at least* and *at most* are taken to be focus-sensitive. They therefore interact with both the ordinary semantic value of their prejacent $[\![\alpha]\!]^O$ and the alternative semantic value of their prejacent $[\![\alpha]\!]^A$. My proposal is that a modal flattens everything it sees: it turns both $[\![\alpha]\!]^O$ and $[\![\alpha]\!]^A$ into singleton sets. Furthermore, I propose that this flattening mechanism is optional. This optionality is the key to deriving both the variation and the epistemic reading with only one scope configuration, which is necessary given that only one scope configuration is available with existential modals.[9]

When the modal takes scope under the modified numeral, as it usually does, its prejacent will not be inquisitive, because it is the numeral modifier that makes the proposition inquisitive. Therefore, flattening $[\![\alpha]\!]^O$ is vacuous in this case. Flattening $[\![\alpha]\!]^A$, on the other hand, will have an effect: it modifies the alternatives the modified numeral can use when it is merged. As we will see in the next section, there are some cases where universal modals take scope over modified numerals. In these cases, flattening $[\![\alpha]\!]^O$ will have an effect: the numeral modifier will have made the prejacent of the modal inquisitive, and flattening undoes this action. In this case, flattening $[\![\alpha]\!]^A$ will be vacuous. When the modal is merged, the numeral modifier has already used the alternatives in its computation, and therefore it no longer matters what the set of alternatives looks like. This is summarised in (78).

(78) **Proposal**
Modals optionally flatten both $[\![\alpha]\!]^O$ and $[\![\alpha]\!]^A$

a. When a modal takes scope under the modified numeral, flattening $[\![\alpha]\!]^O$ is vacuous (because its prejacent will already be flat) but flattening $[\![\alpha]\!]^A$ has an effect

b. When a universal modal takes scope over the modified numeral, flattening $[\![\alpha]\!]^A$ is vacuous (because the numeral modifier has already used the alternatives at this point) but flattening $[\![\alpha]\!]^O$ has an effect

I realise that this discussion is rather dry and perhaps difficult to follow without any examples. I hope the reader will permit me to make two more remarks on the technical mechanism before moving on to the examples. First, I use Beaver & Clark's (2008) theory of focus. Beaver and Clark assume that an

[9] Although I say here that it is the modal that optionally does the flattening, I believe that this cannot be true. The reason is that we would need two lexical entries for each modal in order for it to work: one that flattens and one that does not. A better way to think about it is that there is a flattening operator that is optionally present in the structure. When it is present, it needs to be licensed by a modal.

alternative semantic value $[\![\alpha]\!]^A$ is calculated à la Rooth (1985, 1992) but that a focus-sensitive operator does not interact directly with this set of alternatives. Instead, it interacts with it through the Current Question under discussion CQ. Although the reality is slightly more complex then this, for our purposes here it suffices to say that the CQ must be a subset of $[\![\alpha]\!]^A$. The ordered set of alternatives S that we have already come across is an ordered version of the CQ.

Second, the set S is thus derived from the set of Rooth-Hamblin alternatives. It is this set that *at least* and *at most* use to set a lower bound and an upper bound respectively. MAX_{AL}, on the other hand, uses another ordered set of alternatives AL. This set is not generated via the Rooth-Hamblin alternatives but is a separate set. As the Rooth-Hamblin alternatives can be flattened by modals, so can the CQ and S, which are derived from the Rooth-Hamblin alternatives. AL, on the other hand, exists as a separate entity that is unaffected by such flattening operations. The intuition behind this idea is that although a set consisting of multiple possibilities is sometimes flattened into a single set, this does not mean that there are no alternatives left to the proposition that is being uttered. Regardless of what happens during a particular computation, there can always be alternatives to any proposition. For instance, when we use Kratzer and Shimoyama's (2002) operation of Existential Closure, which puts all the disjuncts of a disjunction into one possibility (i.e. it flattens the disjuncts), we do not want to say that the disjuncts no longer give rise to two (or more) alternative possibilities in the minds of the speakers.

Similarly, if we flatten the possibilities in a sentence with the German free choice indefinite *irgendein* in (79), also from Kratzer and Shimoyama (2002), the modal may flatten the alternatives (of the form *Marry marries doctor x, Mary marries doctor y, ...*) but we still need alternatives to (79), for instance to calculate the run-of-the-mill quantity implicature *Mary does not have to marry all doctors*. Under the present assumptions, S could be flattened by the modal while AL would remain intact for the computation of quantity implicatures.

(79) Mary muss irgendeinen Arzt heiraten.
Mary must IRGENDEIN doctor marry.
'Mary has to marry a doctor, any doctor is a permitted option.'

Finally, if we have a sentence like (80) and the modal flattens the alternatives, which, as we will see, are of the form *Robin and Cormoran solved* n *crimes*), this does not mean that potential alternative numbers of crimes are suddenly not relevant any more for the speaker.

(80) Robin and Cormoran were required to solve at least three crimes.

Using a sentence with a numeral always involves reasoning about numbers and possible alternative numbers and a scale is needed for this. To this end, the scale AL is independent of the set $[\![\alpha]\!]^A$ and is therefore immune to any operations in the computation that might affect the alternatives. This distinguishes it from S, which has no such immunity. Below it will become clear why MAX_{AL} uses

AL rather than S.

3.4 Natural combinations

Now that we are equipped with all this technical paraphernalia, let us return to the data. We will first go through the two combinations that are intuitively the most felicitous ones: *at least* with a universal modal and *at most* with an existential modal. We begin with (3).

(3) Jane is required to invite at least two friends.

Let us first see what reading we derive when the modal does not do any flattening. As mentioned before, I assume wide scope readings for the numeral modifier throughout this section. The prejacent of *at least* is then as in (81).

(81) $[\![\text{Jane is required to invite at least two friends}]\!]^O = \{\Box p_2\}$

The relevant alternatives are is given in (82).

(82) $[\![\text{Jane is required to invite at least two friends}]\!]^A = \{\Box p_0, \Box p_1, \Box p_2, \Box p_3, ...\}$

As explained above, for my purposes it suffices that the CQ be a subset of the set of Rooth-Hamblin alternatives. In this case and all other cases in this paper, the CQ is equivalent to the set of alternatives $[\![\alpha]\!]^A$.

(83) $CQ = [\![\text{Jane is required to invite at least two friends}]\!]^A = \{\Box p_0, \Box p_1, \Box p_2, \Box p_3, ...\}$

The alternatives are ordered as in (84), which is an entailment ordering: if you invite two or more friends in every world, you also invite one or more friends in every world. As before, AL is equivalent to S.

(84) $S = AL = \Box p_0 < \Box p_1 < \Box p_2 < \Box p_3 < \Box p_4 ...$

Applying *at least* to (81) yields (85).

(85) $\{\Box p_2 \wedge \neg\Box p_3,$
$\cup \{\Box p_3 \wedge \neg\Box p_4, \Box p_4 \wedge \neg\Box p_5, ...\}\}$

There are two possibilities in this proposition. The first possibility, $\Box p_2 \wedge \neg\Box p_3$, is obtained simply by applying MAX_{AL} to $\Box p_2$. The set $\{\Box p_3 \wedge \neg\Box p_4, \Box p_4 \wedge \neg\Box p_5, ...\}$ is obtained by applying MAX_{AL} to all higher alternatives, and this set of possibilities is turned into a set of worlds by applying the union operation. $\Box p_2 \wedge \neg\Box p_3$ says that Jane invites two or more friends in every world, but she does not invite three or more friends in every world. In other words: two friends is sufficient; three friends is not required. The other alternatives have the same meaning except with higher numbers. There are two possibilities, so the proposition is inquisitive. This means that epistemic inferences are derived: the speaker is not sure if Jane has to invite two friends and for all numbers above two, the speaker is also not sure that Jane has to invite that many friends. This corresponds to the epistemic reading of (85).

Note that $\cup\{\Box p_3 \wedge \neg\Box p_4, \Box p_4 \wedge \neg\Box p_5, ...\}$ is not equivalent to $\Box p_3$. This is because there must be some number n, somewhere down the line, for which, $\neg\Box p_n$ holds. This is incompatible with $\Box p_3$, which includes all numbers from three to infinity. The fact that there is some number n such that $\neg\Box p_n$ means that there is a limit to what is required; it is not the case that Jane has to invite an infinite number of friends. There is some number n such that n is a sufficient number of friends for Jane to invite.

Now let us see what happens when the modal flattens everything it sees. The prejacent of the modal is p_2; the proposition that Jane invites two friends. The prejacent of the modal is given in (86).

(86) $[\![\text{Jane invites two friends}]\!]^O = \{p_2\}$

This is not an inquisitive proposition, so $[\![\alpha]\!]^O$ is already a singleton set: there is nothing there for the modal to flatten. But assuming that the numeral is the focused element in the sentence, $[\![\alpha]\!]^A$ contains multiple possibilities, as shown in (87).

(87) $[\![\text{Jane invites two friends}]\!]^A = \{p_0, p_1, p_2, ...\}$

The modal flattens this set, as in (88).

(88) $[\![\text{Jane invites two friends}]\!]^A = \{\{w_{[0]}, w_{[1]}, w_{[2]}, ...\}\} = \{p_0\}$

Adding the lexical meaning of the modal yields the ordinary meaning in (89) and the alternatives in (90).

(89) $[\![\text{Jane is required to invite two friends}]\!]^O = \{\Box p_2\}$

(90) $[\![\text{Jane is required to invite two friends}]\!]^A = \{\Box p_0\}$

The CQ is a subset of $[\![\alpha]\!]^A$. Given that $[\![\alpha]\!]^A$ only contains one element, the CQ is now equivalent to it, as shown below.

(91) $\text{CQ} \subseteq [\![\alpha]\!]^A = \{\Box p_0\}$

The set AL, on the other hand, is independent from $[\![\alpha]\!]^A$ and therefore stays as it is. Thus, we have:

(92) $S = \Box p_0$

(93) $AL = \Box p_0 < \Box p_1 < \Box p_2 < \Box p_3 < \Box p_4$

Now we are ready to add *at least*. The meaning of (3) with a flattened set of alternatives is given in (94).

(94) $\{\Box p_2 \wedge \neg\Box p_3\}$

This meaning comes about as follows. First, we apply MAX_{AL} to the prejacent $\Box p_2$. This yields $\Box p_2 \wedge \neg\Box p_3$. Then we take all higher alternatives in S and apply MAX_{AL} to them. But the modal has thrown all higher alternatives of S in the bin. We only have $\Box p_0$ left, which is ranked lower than $\Box p_3$. Furthermore, even if we had higher alternatives, $\Box p_2$ is no longer in the set of alternatives, so the part $p' >_S p$ in the definition of *at least* is vacuous; there is no longer a p to compare p' to. Thus, we only derive the possibility $\Box p_2 \wedge \neg\Box p_3$. This is where we need a separate scale for $\textsc{max}_{AL}$. If $\textsc{max}_{AL}$ used S, it would be unable to apply to the prejacent because there is no prejacent left in S. The fact that $\textsc{max}_{AL}$ uses AL enables it to yield (94) even when S has been flattened to contain only $\Box p_0$.

The proposition in (94) is not inquisitive, so we do not derive epistemic implicatures. The meaning is that Jane is required to invite two or more friends but she is not required to invite three or more friends. This is the authoritative reading with a variation inference. She has to invite at least two friends, and she is free to choose a number of friends to invite in the [3-∞) range. In other words, it is not the case that there is some number higher than the number two such that she has to invite that number of friends. So, when the modal does not do any flattering, we derive the epistemic reading. When the modal does flatten, we get the authoritative reading.

Now let us move on to the other natural combination: *at most* with *allowed*, exemplified in (2).

(2) Jane is allowed to invite at most two friends.

S is now as in (95), which is again an entailment ordering.

(95) $S = AL = \Diamond p_0 < \Diamond p_1 < \Diamond p_2 < \Diamond p_3 < \Diamond p_4$ ...

Let us first assume that the modal does not do anything except add its regular meaning to the mix, as in (96).

(96) $[\![\text{Jane is allowed to invite two friends}]\!]^O = \{\Diamond p_2\}$

Merging *at most* results in the meaning in (97).

(97) $\{\Diamond p_2 \wedge \neg\Diamond p_3,$
$\cup \{\Diamond p_0 \wedge \neg\Diamond p_1, \Diamond p_1 \wedge \neg\Diamond p_2\}\}$

$\textsc{max}_{AL}$ $\Diamond p_2$ yields the first possibility. The second possibility is the union of the possibilities $\textsc{max}_{AL}$ $\Diamond p_n$ for all numbers lower than 2: $\Diamond p_0$ and $\Diamond p_1$. The speaker conveys two possibilities: Jane is allowed to invite two friends but no more or she is allowed to invite fewer than two friends but no more. In other words: the upper bound is two or it is lower than two. The fact that the proposition is inquisitive means that an epistemic implicature can be calculated: the speaker does not know whether the upper bound is two or some number under two. This corresponds to the epistemic reading.

Now we will consider the other reading of the sentence, where the modal puts all possibilities in the alternatives into a single possibility. We start off with the sister of the modal, given in (98).

(98) $[\![\text{Jane invites two friends}]\!]^O = \{p_2\}$

The alternatives the modal gets are the ones in (99), and flattening this set yields (100).

(99) $[\![\text{Jane invites two friends}]\!]^A = \{p_0, p_1, p_2, ...\}$

(100) $[\![\text{Jane invites two friends}]\!]^A = \cup\{p_0, p_1, p_2, ...\} = \{p_0\}$

Adding the modal gives us (101) and (102).

(101) $[\![\text{Jane is allowed to invite two friends}]\!]^O = \{\Diamond p_2\}$

(102) $[\![\text{Jane is allowed to invite two friends}]\!]^A = \{\Diamond p_0\}$

Given that S in an ordered version of the CQ and CQ $\in [\![\alpha]\!]^A$, we get (103). AL, being independent of $[\![\alpha]\!]^A$, remains unaffected by this change, as shown in (104).

(103) $S = \Diamond p_0$

(104) $AL = \Diamond p_0 < \Diamond p_1 < \Diamond p_2 < \Diamond p_3 < \Diamond p_4$

After adding *at most*, we get the final meaning in (105).

(105) $\{\Diamond p_2 \wedge \neg\Diamond p_3\}$

As in the *at least* case, this is simply MAX_{AL} applied to the prejacent. This is possible because while S has been flattened, AL is still as in (95). As for the other alternatives, there is one alternative that is lower than $\Diamond p_2$, namely $\Diamond p_0$, the only alternative we have left. But according to the definition of *at most*, we have to find all $p' <_S p$. p is the prejacent $\Diamond p_2$, but $\Diamond p_2$ has been taken out of CQ and is therefore no longer ordered by S. As a result, we still cannot pick out any alternative, and are left with just the first possibility.

(105) says that Jane is allowed to invite two friends but she is not allowed to invite three friends. Thus, it places an upper bound of two on the number of friends Jane is allowed to invite. There is no epistemic implicature because there is only one possibility. This is the authoritative reading we wanted to derive.

In sum, we derive both an authoritative reading and an epistemic reading for the two natural combinations, as desired.

3.5 Less natural combinations

I will discuss the less natural readings in a slightly different order. First I will show how an epistemic reading can be derived for both combinations, and then I will turn to the variation readings.

(5) is the least natural combination, and it only has an epistemic reading.

(5) Jane is allowed to invite at least two friends.

The ordered alternatives are as in (95) and the denotation is given in (106).

(52) $S = AL = \Diamond p_0 < \Diamond p_1 < \Diamond p_2 < \Diamond p_3 < \Diamond p_4$ ...

(106) $\{\Diamond p_2 \wedge \neg\Diamond p_3,$
$\cup \{\Diamond p_3 \wedge \neg\Diamond p_4, \Diamond p_4 \wedge \neg\Diamond p_5, ...\}\}$

As before, the first possibility is simply $\textsc{max}_{AL}$ $\Diamond p_2$. The second possibility is the result of applying $\textsc{max}_{AL}$ to the higher alternatives and taking the union of the resulting set of possibilities. The first possibility says that Jane is allowed to invite two but not three friends. The second says that she is allowed to invite three friends but there is some number for which she is not allowed to invite that many friends. The second possibility is equivalent to (107).

(107) $\Diamond p_3 \wedge \exists p'[p' >_S p_3 \wedge \neg\Diamond p']$

Together, these possibilities say that either Jane is allowed to invite two friends but no more, or she is allowed to invite some other number of friends above two, but there is an upper bound to how many friends she is allowed to invite. The proposition is inquisitive so the reading is epistemic. Recall from section 3.3 that sentences with *at least* and an existential modal only have the stronger epistemic reading that conveys ignorance about where the upper bound to what is allowed is and not the weaker epistemic reading that merely conveys ignorance about which numbers are allowed. The reading I have derived here is thus precisely the kind of epistemic reading we want: it is not merely an ignorance reading but more specifically ignorance about where the upper bound is.

We now turn to our final combination in (4), with the ordered alternatives in (108).

(4) Jane is required to invite at most two friends.

(108) $S = AL = \Box p_0 < \Box p_1 < \Box p_2 < \Box p_3 < \Box p_4$ ...

The reading we derive is shown in (109). The possibilities are derived as usual: by first applying $\textsc{max}_{AL}$ to the prejacent of *at most* and then to the alternatives that are ordered lower, after which we take the union of the lower alternatives.

(109) $\{\Box p_2 \wedge \neg\Box p_3,$
$\cup \{\Box p_0 \wedge \neg\Box p_1, \Box p_1 \wedge \neg\Box p_2\}\}$

The first possibility is that Jane has to invite two friends but she need not invite more than two friends. The second possibility is that there is some number below the number two such that she has to invite that many friends (this number can also be zero). This possibility is equivalent to (110) (which is equivalent to its second conjunct, given that $\Box p_0$ is a tautology). There are two possibilities, so the reading is an epistemic one. The speaker conveys that either Jane has to invite at least two friends or she has to invite some minimum number of friends below two. There is ignorance about the lower bound. This is the epistemic reading we wanted to derive.

(110) $\Box p_0 \wedge \neg\exists p'[p' <_S p_2 \wedge \Box p']$

Now let us try to derive variation readings for 3.5 and (107), starting with 3.5. Flattening gives us a CQ that only contains $\Diamond p_0$, as before. And like before, using AL we only derive $\textsc{max}_{AL}\ \Diamond p_2$, because there are no alternatives p' left in S such that $p' >_S \Diamond p_2$. We derive (111).

(111) $\{\Diamond p_2 \wedge \neg\Diamond p_3\}$

This is clearly not a possible reading of 3.5; it says that Jane is allowed to invite *at most* two friends. In fact, it is equivalent to the authoritative reading of (2), in (105). Before I say more about this, let us have a look at the flattened reading of (107).

When the alternatives are flattened, we are left with a CQ containing only $\Box p_0$ again. Given that the prejacent $\Box p_2$ is no longer in the set of alternatives, the part $p' <_S p$ of the denotation of *at most* cannot pick out any possibilities. We only derive $\textsc{max}_{AL}\ \Box p_2$, which is equivalent to (112).

(112) $\{\Box p_2 \wedge \neg\Box p_3\}$

This is not an attested reading of (4). It is a reading that sets a lower bound: Jane must invite at least two friends but she need not invite more. This reading is equivalent to the variation reading of (3) in (94).

So, (111) is equivalent to (105) and (112) is equivalent to (94). This can also be seen in the table below, where the readings with a # refer to derived but unattested readings. In the two 'authoritative' rows, *at least* and *at most* yield the same denotation.

		at least	***at most***
$\Diamond$	aut.	(111) $\#\{\Diamond p_2 \wedge \neg\Diamond p_3\}$	(105) $\{\Diamond p_2 \wedge \neg\Diamond p_3\}$
	ep.	(106) $\{\Diamond p_2 \wedge \neg\Diamond p_3,$ $\cup\{\Diamond p_3 \wedge \neg\Diamond p_4, \Diamond p_3 \wedge \neg\Diamond p_4, ...\}\}$	(97) $\{\Diamond p_2 \wedge \neg\Diamond p_3,$ $\cup\{\Diamond p_0 \wedge \neg\Diamond p_1, \Diamond p_1 \wedge \neg\Diamond p_2\}\}$
$\Box$	aut.	(94) $\{\Box p_2 \wedge \neg\Box p_3\}$	(112) $\#\{\Box p_2 \wedge \neg\Box p_3\}$
	ep.	(85) $\{\Box p_2 \wedge \neg\Box p_3,$ $\cup\{\Box p_3 \wedge \neg\Box p_4, \Box p_4 \wedge \neg\Box p_5, ...\}\}$	(109) $\{\Box p_2 \wedge \neg\Box p_3,$ $\cup\{\Box p_0 \wedge \neg\Box p_1, \Box p_1 \wedge \neg\Box p_2\}\}$

Table 1: Summary of denotations

In table 2 I have summarised the types of readings this analysis derives for each combination, again with # signifying unattested readings.

		at least	***at most***
$\Diamond$	authoritative epistemic	#UB LB, UB ignorance	UB UB, UB ignorance
$\Box$	authoritative epistemic	LB LB, LB ignorance	#LB UB, LB ignorance

Table 2: Summary of readings

As this table shows, the epistemic readings have two kinds of bounds: the bound set by the lexical item (a lower bound for *at least* and an upper bound for *at most*) and the bound that the epistemic inference is about, set by the modal (a lower bound for a universal modal and an upper bound for an existential modal). Below I will go through each combination.

For the natural combination $\Box$ + *at least*, the account derives an authoritative reading that sets a lower bound. It also derives an epistemic reading that sets a lower bound, and the ignorance on the part of the speaker is also about where lower bound is. That is, for (3), repeated below, the epistemic reading is that there is some lower bound of friends Jane needs to invite, and the speaker does now know whether this lower bound is two or higher.

(3) Jane is required to invite at least two friends.

Similarly, for our other natural combination $\Diamond$ + *at most*, the authoritative reading sets an upper bound, and the epistemic reading conveys ignorance about the upper bound. For (2), there is some upper limit to the number of friends Jane can invite, and this upper limit is two or lower.

(2) Jane is allowed to invite at most two friends.

For these two natural combinations, then, the bound set by the modal corresponds to the bound set by the numeral modifier.

For (5), which exemplifies the less natural $\Diamond$ + *at least* combination, the epistemic reading is that there is some upper bound to the number of friends Jane is allowed to invite, and this upper bound is at least two. Thus, the ignorance is about where the upper bound is, but the sentence still conveys a lower bound: the upper bound is two or *higher* than two.

(5) Jane is allowed to invite at least two friends.

The other less natural combination (4) works the same way, except that the bounds are now flipped. In the epistemic reading, the ignorance is about the lower bound to what is required, but there is an upper bound to this lower bound: the lower bound is *no higher* than two.

(4) Jane is required to invite at most two friends.

As mentioned above, the modal determines the kind of bound the speaker is ignorant about, with $\Diamond$ yielding a lower bound and $\Box$ yielding an upper bound. The numeral modifier determines the bound of the variation reading and the 'bound of the bound' of the ignorance reading. For instance, using a universal modal as in (107) means that the ignorance is about the lower bound, but adding *at most* to the mix sets an upper bound to where this lower bound can be.

In all the cases I have discussed so far, the bound set by the numeral modifier is maintained. So, even though the 'ignorance bound' contributed by the modal may not be the same as the bound set by the numeral modifier, the numeral modifier still contributes a limit to this 'ignorance bound'. In all of these cases, *at least* contributes a lower bound and *at most* contributes an upper bound.

Now let us consider the non-attested variation readings we have derived for the $\Diamond$ + *at least* combination and the $\Box$ + *at most* combination. Here we see that the bound of the modal has prevailed, and there is nothing left of the bound of the modified numeral. The *at least* example sets an upper bound and the *at most* example sets a lower bound. I propose that this is why these readings are not attested. The primary meaning contribution of *at least* and *at most* is to set a bound, and in (111) and (112) this bound has completely disappeared.

One way to explain this by saying that there is a principle in language that states that when a lexical item contributes a meaning, this meaning must be maintained throughout the rest of the derivation. This is reminiscent of Buccola and Spector's (2016:165) 'Pragmatic economy constraint' on numerals. This constraint says that a sentence with a numeral n is infelicitous if replacing this numeral by a different numeral m would result in the same meaning. Minimally rephrasing their constraint for our current purposes as in (113) would not work, however:

(113) **Pragmatic economy constraint** (non-final)
An LF ϕ containing a numeral modifier M is infelicitous if, for some N distinct from M, ϕ is truth-conditionally equivalent to $\phi[M \mapsto N]$

The reason this does not work is because it would rule out both denotations on the second and the fourth row of table 1 instead of only the denotations carrying a #. In other words, if the constraint in (113) held, we would expect *all* authoritative readings ((111), (105), (94), (112)) to be bad. This indicates that we must be more specific about which numeral modifier survives and which is ruled out. This is done by (114).

(114) **Pragmatic economy constraint**
For lower-bounded and upper-bounded numeral modifiers M, an LF ϕ containing M is only felicitous if ϕ sets the same bound as M.

This constraint correctly rules out (111) and (112) but not (105) and (94). (114) can be viewed as an economy principle: it is not efficient to use an expression with a certain meaning (in particular: a lower bound or upper bound) only to subsequently remove this meaning in the computation.

Another way to think about the reason why (111) and (112) are not attested is to invoke a blocking mechanism. Nouwen (2010) also uses such a mechanism to block certain readings with class B modifiers. His way of implementing the notion of blocking is that whenever a marked form and an unmarked form convey the same meaning, the unmarked form is given precedence, and the marked form is blocked from having this meaning. For him, the competition is between class B modifiers and bare numerals, with the bare numeral denotations being less marked. In this case, we also have two cases where the meanings are identical: (111) is identical to (105) and (112) is identical to (94). But (105) and (94) get their meaning in a less convoluted way. In these cases, the bound set by the numeral modifier corresponds to the bound set by the modal. In the case of (111) and (112), the derivation involves a reversal of the bound. This is not quite the same as the notion of marked versus unmarked meanings. Instead, the difference is that in one case, the derivation involves the rather complex and counterintuitive step of turning a lower bound into an upper bound or vice versa, while the other derivation does not. The simpler derivation is preferred. This could be a constraint on language proper in the sense that it is not only forms but also derivations that compete with one another. This would need to be made more precise, because in the mechanism as I have proposed it there is not a single step we can point at that is more complex than some step in the derivation that is not blocked. It could also be a cognitive constraint: it is cognitively costly to use an expression that sets a lower bound in a proposition that sets an upper bound or vice versa. This could be because it requires changing the meaning from one bound to the other mid-computation.

Thus, the idea is that it is not economical to use an expression with a certain meaning component and to then delete this meaning component in the derivation, meaning that there is nothing left of it in the proposition we derive.

4 Benefits of the account

As shown in the previous section, the present account is the first that derives the observed readings for all combinations of modals and modified numerals. Here I discuss three further benefits of the account. First I show that the uncommon but possible authoritative readings of sentences with *at most* and a universal modal can still be derived. Then I demonstrate that this account is also the first that can derive ignorance and 'authoritative' (variation) readings of sentences with universal nominal quantifiers and numeral modifiers. Finally I discuss the merits of using this particular mechanism in inquisitive semantics to derive ignorance readings.

4.1 Authoritative readings with *at most* and universal modals

In this section I will return to example (4). I have called this one of the less natural combinations and said that we can only derive an epistemic reading for it.

(4) Jane is required to invite at most two friends.

It is clear that (2) is a better candidate for expressing that Jane is not allowed to invite more than two friends.

(2) Jane is allowed to invite at most two friends.

However, (4) still does have an authoritative reading. So far I have not derived this reading. It turns out that this is actually the surface scope reading of (4), with the structure in (115).

(115) [$\Box$ [Jane invites [at most [[2 *many*] friends]]]]

Recall that this sentence can have a surface scope structure because nothing stops modified numerals from taking scope under universal modals.

The prejacent of *at most*, with focus on the numeral, gives rise to the Rooth-Hamblin alternatives in (116).

(116) $[\![\text{Jane invites [two]}_F \text{ friends}]\!]^A = \{p_0, p_1, p_2, p_3...\}$

As usual, the CQ is derived from this set and is ordered as S in (117), and we have an equivalent AL.

(117) $S = AL = p_0 < p_1 < p_2 < p_3 < p_4$ ...

Adding *at most*, we derive (118) as the meaning of the prejacent of the modal.

(118) $[\![\text{Jane invites at most two friends}]\!]^O = \{p_2 \wedge \neg p_3, \{w_{[0]}, w_{[1]}\}\}$

This is an inquisitive proposition containing two possibilities: the possibility that Jane invites exactly two friends and the possibility that she invites fewer than two friends.

We saw above that a modal optionally flattens both the ordinary semantic value and the alternative semantic value of its prejacent. In this case, flattening $[\![\alpha]\!]^A$ will not do much, because there is no operator above the modal that needs to use the alternatives. *At most* has already done this below the modal. Flattening $[\![\alpha]\!]^O$, on the other hand, does have an effect. As shown in (119), the modal now turns the inquisitive proposition in (118) into a non-inquisitive proposition containing only one possibility.

(119) $\{\Box \cup \{p_2 \wedge \neg p_3, \{w_{[0]}, w_{[1]}\}\}\} = \{\Box\{w_{[0]}, w_{[1]}, w_{[2]}\}\}$

(119) says that in all accessible worlds, Jane invites between zero and two friends and no more. This is the authoritative reading of (107).

Note that it is also possible, though not necessary, to derive an additional authoritative reading with a universal modal and *at least* this way. For (3), the prejacent of the modal is as in (120).

(36) Jane is required to invite at least two friends.

(120) $\{p_2 \wedge \neg p_3, p_3\}$

When the modal flattens (120), we get (121). This says that in all worlds, Jane invites two or more friends.

(121) $\{\Box \cup \{p_2 \wedge \neg p_3, p_3\}\} = \{\Box p_2\}$

Thus, the surface scope configuration that is available when modified numerals occur with universal modals allows for the generation of an authoritative reading with *at most*, which we indeed observe. It also enables the calculation of a harmless extra authoritative reading with *at least*.

4.2 Modified numerals and universal quantifiers

Sentences with universal modals and class B numeral modifiers, such as (122), also have two readings.

(122) Everyone adopted at least two cats.

The most obvious reading is a non-epistemic reading: everyone adopted two cats or more. In parallel to the authoritative readings we have seen, this reading comes with a variation inference: not everyone adopted the same number of cats. That is, (122) is infelicitous when the speaker knows that everyone adopted, say, exactly ten cats. There is also an epistemic reading, which is a bit more difficult to get: everyone adopted the same number of cats, and this number is two or higher. We know that *at least* cannot move over *everyone* by itself since this instantiates a violation of the Heim-Kennedy Generalisation (Kennedy, 1997; Heim, 2000). Therefore, (122) represents another case where we only get one scope configuration but we do observe two readings, just like the cases where numeral modifiers must take scope over existential modals.

Using the analysis laid out above, we can actually derive both readings from the surface scope configuration. The only assumption we need, which may or may not be a slightly controversial one, is that universal quantifiers also have the ability to optionally flatten their prejacent. Let us go through the derivation of (122) to see how this works. The denotation of the prejacent of *everyone* is given in (123).

(123) $\{\lambda x \,.\, \text{MAX}_{AL}[\exists y[\#y = 2 \wedge \text{cats}(y) \wedge \text{adopts}(x,y)]] \,,\, \lambda x \,.\, \cup \{\, \text{MAX}_{AL}\; p' \mid p' >_S [\exists y[\#y = 2 \wedge \text{cats}(y) \wedge \text{adopts}(x,y)]]\}\}$

This meaning comes about through the use of the denotation of *at least* in (60) where α is $\langle\langle e,p\rangle,p\rangle$ and β is $\langle e,p\rangle$, as in (64).

(23) $[\![\text{at least}]\!]^S = \{\lambda\alpha\lambda\beta.\text{MAX}_{AL}\;(\alpha(\beta))\,,\, \lambda\alpha\lambda\beta.\cup\{\text{MAX}_{AL}\; p' \mid p' \;>_S \alpha(\beta)\}\}$

(27) $[\![\text{at least}]\!]^S = \{\lambda\mathcal{P}_{\langle\langle e,p\rangle,p\rangle}\lambda Q_{\langle e,p\rangle}.\text{MAX}_{AL}\;(\mathcal{P}(Q))\;,\; \lambda\mathcal{P}_{\langle\langle e,p\rangle,p\rangle}\lambda Q_{\langle e,p\rangle}.\cup\{\text{MAX}_{AL}\; p' \mid p' \;>_S \mathcal{P}(Q)\}\}$

(123) gives us two possibilities: the *exactly 2* possibility and the *3 or more* possibility. Adding the universal quantifier yields (124).

(124) $\{\forall x\; [\;\text{MAX}_{AL}[\exists y[\#y = 2 \wedge \text{cats}(y) \wedge \text{adopts}(x,y)]]]\;,\; \forall x\;[\;\cup\;\{\;\text{MAX}_{AL}\; p' \mid p' >_S [\exists y[\#y = 2 \wedge \text{cats}(y) \wedge \text{adopts}(x,y)]]]\}\}$

If P_n stands for 'the set of people who adopted n cats', the meaning in (124) can be represented as in (125).

(125) $\{\forall x P_2 \wedge \forall x \neg P_3 \;,\; \cup\{\forall x P_3 \wedge \forall x \neg P_4 \;,\; \forall x P_4 \wedge \forall x \neg P_5, ...\}\}$

The first possibility is the possibility that everyone adopted two cats and no more. The second possibility is the union of all possibilities such that everyone adopted a higher number than two cats n and no-one adopted more than n cats.

Thus, either everybody adopted exactly two cats or everybody adopted some higher number of cats. Either way, everyone adopted the same number of cats. This proposition is inquisitive, so we derive the inference that the speaker does not know which possibility is true. This is the epistemic reading of (122).

To derive the non-epistemic reading, the universal quantifier must flatten (123). Note that this is a case of the relevant operator flattening $[\![\alpha]\!]^O$ rather than $[\![\alpha]\!]^A$, as in section **??** (i.e. this corresponds to the mechanism described in (78-b) except that a universal quantifier is used instead of a universal modal).

The union of the two sets in (123) is given in (126): *exactly 2* or *3 or more* comes down to *at least 2*.

(126) $\{\lambda x.\exists y[\#y = 2 \wedge \text{cats}(y) \wedge \text{adopts}(x, y)]\}$

Adding in the universal quantifier, the proposition we end up with is the one in (127).

(127) $\{\forall x.\exists y[\#y = 2 \wedge \text{cats}(y) \wedge \text{adopts}(x, y)]\}$

This is a singleton set, so it is not inquisitive. It says that for everyone x there is a certain number of cats y that x adopts, and the cardinality of y is 2. Given the one-sided meaning of the bare numeral 2, (127) has a lower-bounded reading: everyone adopted two or more cats. This is the most prominent reading of (122) without an epistemic implicature.

Thus, the optional flattening mechanism can also derive the two readings we observe with universal quantifiers, without relying on any non-existent scope configurations. As degree-based accounts of modified numerals do use two scope configurations to get two readings, the Heim-Kennedy Generalisation prevents them from predicting the attested ambiguity in cases with nominal quantifiers. My account solves this issue.

4.3 A note on the nature of epistemic inferences

Finally, I will make some observations about the exact nature of the epistemic inferences we have derived. Coppock and Brochhagen (2013), who were the first to use inquisitive semantics in an account of modified numerals, derived meanings of the form in (129) for sentences like (128).

(128) Malika adopted at least two cats.

(129) $\{p_2, p_3, p_4, p_5, ...\}$

Thus, for each number from two upwards, the proposition contains the possibility that Malika adopted that number of cats.

Ignorance was derived as a Quality implicature, but in a slightly different way than I have done here. Coppock & Brochhagen posited the so-called *Maxim of Interactive Sincerity*. This Maxim said that when a proposition is interactive, it must also be interactive in the speaker's information state. Here interactivity means 'containing multiple possibilities'.[10] Informally, when you utter a proposition with different possibilities, those possibilities must be possibilities in your mind.

Schwarz (2016) pointed out that there is a problem with this way of deriving epistemic inferences: you cannot utter (128) unless you consider it a possibility that Malika adopted exactly two cats. In general, the numeral modified by *at least* or *at most* must always correspond to one of the possibilities the speaker considers. Coppock and Brochhagen do not derive this. Say that you think that Malika adopted either nine or ten cats, and you utter (128). Coppock & Brochhagen predict that this is felicitous. After all, the number is two or higher and the proposition is interactive in your information state.

This shows that the epistemic inference Coppock & Brochhagen derive is too weak in general. For instance, according to their account, you can say *at least one* when you know that the actual number is either one million or two million, and you can say *at most a thousand* when you are unsure whether the number is one or two.

They could remedy this by saying that the possibilities in the speaker's information state must be equivalent to the possibilities in the proposition. A speaker who utters (128) must then consider all the possibilities in (129) to be potentially true. But now the epistemic reading is too strong. Say that there is a particular shelter where you can only adopt pairs of two cats. Then a speaker who knows this and who also knows that Malika adopted a certain number of cats from this shelter can felicitously utter (128) even though all possibilities p_n where n is an odd number are not in the speaker's information state.

Ciardelli et al. (2016), inspired by Quantity implicature-based accounts such as Büring (2008), Schwarz (2013), and Kennedy (2015), solve this problem by saying that (128) denotes the possibilities in (130): either Malika adopted exactly two cats or she adopted some number of cats above two.

(130) $\{p_2 \wedge \neg p_3, p_3\}$

Now we can say that the possibilities in the speaker's information state must correspond to the possibilities in the proposition and generate the right implicatures that way. Either the number in the prejacent of the modified numeral is the right number or it is some number higher than that number, but not all higher numbers have to be live possibilities for the speaker. I have followed Ciardelli et al. (2016) in adopting this method, and therefore my analysis, too, generates the right kinds of implicatures.

As a final remark, quality implicatures are more difficult to cancel than quantity implicatures, as mentioned by Ciardelli et al. (2016). As can be observed in (131), this is a correct prediction.[11]

[10] The reason why the terminology is different is because Coppock & Brochhagen use $\mathrm{Inq}_\cup$ instead of Inq_B, which allows nested possibilities. In this framework, an interactive proposition is a proposition that contains multiple possibilities. Interactivity does not necessarily imply inquisitivity, because for a proposition to be inquisitive it has to contain at least two independent possibilities.

[11] But see Alexandropoulou (2018), chapter 5, for a more careful discussion of these data

(131) a. Malika adopted at least two cats. #In fact, she adopted four.
b. Malika adopted more than two cats. In fact, she adopted four.

Adding the information that Malika adopted four cats implies exact knowledge of the number of cats she adopted, and this is incompatible with the epistemic inference of *at least*, making (131-a) infelicitous. On the other hand, it is fine to add this information to a *more than* sentence like in (131-b), which suggests that *more than* either does not give rise to epistemic inferences or gives rise to weaker, perhaps quantity, implicatures.

5 Conclusion

In this paper I presented an account of the behaviour of class B numeral modifiers. The analysis was based on the notion that these numeral modifiers are focus-sensitive operators and interact with the QUD through Beaver & Clark's (2008) focus principe. In addition, an optional flattening mechanism was introduced to account for the ambiguity we see when class B modifiers occur with modals.

This way of analysing the data takes into account the scopal behaviour of these numeral modifiers as well as the fact that certain combinations of modifiers and modals are more natural than others, and these combinations are also more likely to give rise to authoritative readings.

Finally, the account correctly derives a strong semantic upper bound for cases where *at most* combines with an existential modal. Unlike previous accounts, it also captures interactions between class B modifiers and universal nominal quantifiers, which give rise to an ambiguity that is parallel to the ambiguity observed in the modal domain.

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that suggests that these facts might be slightly different.

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